The Perturbation Technique for finding "Marginal" Equations, and other useful things.

1 Perturbation Technique Applied to non-linear Demand Curves

Consider a demand curve of this shape:

Q = a/P

Suppose that we're interested in finding the elasticity of this demand curve. Remember, elasticity is of the form:

$E_d = \frac{P}{Q}\frac{\Delta Q}{\Delta P}$

When we face a linear demand function, it's easy to find $\frac{\Delta Q}{\Delta P}$. However, this is most definitely NOT a linear demand function. So, how can we find it in this case? Here's a technique that will work.

1. Replace each variable X with "X + Δ X" (Note: treat "Δ X" as a DIFFERENT variable than X.)
2. Solve the equation for the ratio you want.
3. If there are stray Δ X when you have the ratio by itself on one side of the equation, act like those are zero. (Since we're usually looking for these things at a point, it makes sense to assume the change to be very small, and there's not much difference between "very small" and "zero".)

So, let's apply the technique here.

Step 1:

Q+ΔQ = a/(P+Δ P)

Step 2:

Lots of algebra here. First, I'm going to get rid of "Q" by substituting in the original equation.

a/P+ΔQ = a/(P+Δ P)

Now, let's get rid of fractions. I hate fractions. So, I need to multiply both sides by P first.

a+PΔQ = (aP)/(P+Δ P)

Now, by (P+Δ P).

(a+PΔQ)(P+Δ P) = aP

Now, let's distribute, because I hate quantities.

aP + P^2ΔQ + aΔ P + PΔQΔ P = aP

Ah, we can cancel the aP, as it appears on both sides.

P^2ΔQ + aΔ P + PΔQΔ P = 0

Now, let's get all of the ΔQ on one side, by itself.

$P^2\Delta$Q + PΔQΔ P = -aΔ P

Now, to get the ratio we want, divide by Δ P

$P^2\frac{\Delta Q}{\Delta P}$ + PΔQ = -a

To make sure that the ratio is totally alone, let's move the other on the left over to the right, and divide by P^2

$\frac{\Delta Q}{\Delta P}$ = -a/P^2 - ΔQ/P

Alright, now we have the ratio all solved. Just one thing left to do... act like ΔQ on the right hand side is zero. Then, we have:

$\frac{\Delta Q}{\Delta P}$ = -a/P^2

Tah dah!

Now, we can plug that into the elasticity equation...

$E_d = \frac{P}{Q}\frac{a}{P^2}$

Now, with a quick substitution for the demand curve...

$E_d = \frac{P}{a/P}\frac{a}{P^2}$

Funny... Let me multiply top and bottom of the first fraction by P...

$E_d = \frac{P^2}{a}\frac{a}{P^2}$

Interesting, it looks like, once we simplify things...

$E_d = 1$

This form gives us a demand curve that is unit elastic, regardless where we are on it! (Also, what "a" is doesn't seem to matter for elasticity!)

Note: this provides the same answer as if we were using calculus to find the same thing, we just get there with algebra.

2 Perturbation applied to Marginal Revenue

We can use the same technique when calculating marginal revenue for a monopolist.

Say that demand is Q = a - bP.

TR = PQ.

MR = $\frac{\Delta TR}{\Delta Q}$

Since marginal revenue requires a change in quantity, it makes sense to rearrange the demand curve so that we can substitute price out of the total revenue equation. So...

P = (a - Q)/b

Now, we can substitute that into the total revenue equation.

TR = (a-Q)/b * Q

Now, using our technique...

Step 1:

TR + Δ TR = (a-(Q+Δ Q))/b * (Q + Δ Q)

Step 2:

Let's get rid of "TR" by substituting...

(a-Q)/b * Q + Δ TR = (a-(Q+Δ Q))/b * (Q + Δ Q)

Now, let's get rid of fractions by multiplying by b.

(a-Q) * Q + b Δ TR = (a-(Q+Δ Q)) * (Q + Δ Q)

Now, let's distribute everything.

aQ - Q^2 + b Δ TR = aQ + a Δ Q - Q^2 - Q Δ Q - Q Δ Q - $(\Delta Q)^2$

Ooo, we can eliminate a couple terms and combine a couple others.

b Δ TR = a Δ Q - 2Q Δ Q - $(\Delta Q)^2$

Looks like we have to divide by b to get Δ TR by itself.

Δ TR = a Δ Q/b - 2Q Δ Q/b - $(\Delta Q)^2$/b

Now, divide by Δ Q.

$\frac{\Delta TR}{\Delta Q}$ = a/b - 2Q/b - Δ Q/b

Got it!

Step 3: Just have that pesky Δ Q to be rid of.

$\frac{\Delta TR}{\Delta Q}$ = a/b - 2Q/b

And that's it!

Look at that equation. Remember it. Why? Because it's a *general form that will usually work*. All you need is a single condition:

Demand is of the form: Q = a - bP

(Note: if demand is of the form: P = a - bQ, then this formula will NOT work. However, you can use the perturbation technique to solve for a formula that will work in that case, if you like. The formula you're looking for is: $\frac{\Delta TR}{\Delta Q}$ = a - 2bQ. See if you can derive it.)

3 Perturbation and Cost

Now, let's try it with marginal cost!

Say we have a cost function that looks like this:

TC = aQ^3 + FC

(FC is a fixed cost - that is it does not change, and should NOT be considered a variable.)

Let's use our technique to find MC = $\frac{\Delta TC}{\Delta Q}$

Step 1:

TC + ΔTC = a$(Q + \Delta Q)^3$ + FC

Remember, FC is NOT a variable, so it doesn't get a Δ term!

Step 2:

Let's do two things: first, I'll substitute for TC. Also, I'll write out the cubic term more extensively.

aQ^3 + FC + ΔTC = a(Q+Δ Q)(Q+Δ Q)(Q+Δ Q) + FC

Ah, we can already get rid of FC - which proves that fixed costs have NO impact on marginal cost. (We could do this in a more general form to prove it more generally, but I'll leave that to you.) Also, let's distribute the first two terms in that nasty thing on the right hand side.

aQ^3 + ΔTC = a(Q^2+Q Δ Q+ Q Δ Q + $(\Delta Q)^2$)(Q+Δ Q)

Now, combine terms in that big parenthetical expression.

aQ^3 + ΔTC = a(Q^2+2Q Δ Q + $(\Delta Q)^2$)(Q+Δ Q)

Now, let's distribute again.

aQ^3 + ΔTC = a(Q^3 + Q^2 Δ Q +2Q^2 Δ Q + 2Q $(\Delta Q)^2$ + Q $(\Delta Q)^2$ + $(\Delta Q)^3$)

Now, put that a through the whole thing and combine terms.

aQ^3 + ΔTC = aQ^3 + 3aQ^2 Δ Q + 3aQ $(\Delta Q)^2$ + a$(\Delta Q)^3$

We get to eliminate another term!

ΔTC = 3aQ^2 Δ Q + 3aQ $(\Delta Q)^2$ + a$(\Delta Q)^3$

Step 3:

$\frac{\Delta TC}{\Delta Q}$ = 3aQ^2 + 3aQ (Δ Q) + a$(\Delta Q)^2$

And then, treat (Δ Q) like zero...

$\frac{\Delta TC}{\Delta Q}$ = 3aQ^2

And we have it!